Phase diagram of frustrated mixed-spin ladders in the strong-coupling limit

S. Chen[a], L. Wang, and Y.P. Wang

Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100080, P.R. China

Abstract. We study the ground-state properties of frustrated Heisenberg ferrimagnetic ladders with anti-ferromagnetic exchange interactions and two types of alternating sublattice spins. In the limit of strong rung couplings, we show that the mixed spin-1/2 and spin-1 ladders can be systematically mapped onto a spin-1/2 Heisenberg model with additional next-nearest-neighbor exchanges. The system is either in a fer-rimagnetic state or in a critical spin-liquid state depending on the competition between the spin exchanges along the legs and the diagonal exchanges.

PACS. 75.10.Jm Quantized spin models

1 Introduction

In the past years, there has been increasing theoretical interest in the quantum ferrimagnetic systems since the experimental realization of bimetallic quasi-one-dimensional (quasi-1D) magnets [1–3]. Pioneering experiments on synthesizing the bimetallic chain compounds with each unit cell containing two spins were carried out successfully by Kahn et al. [1]. Typical compounds include two families of ferrimagnetic chains described by $ACu(pba)(H_2O)_3 \cdot nH_2O$ and $ACu(pbaOH)(H_2O)_3 \cdot nH_2O$, where $pba = 1,3$-propylenebis (Oxamato), $pbaOH =$ 2-hydroxo-1,3-propylenebis (Oxamato) and A = Ni, Fe, Co, and Mn [2,3]. These materials are quasi-1D bimetallic molecular magnets containing two different transition-metal ions per unit cell alternatingly distributed on a chain. Most of these materials are described by Heisenberg mixed-spin models with antiferromagnetic interactions. This has stimulated theoretical studies on the mixed-spin systems. A number of recent studies have been focused on ground-state properties of mixed-spin chain [4–8] and ladders [9–15] composed of two kinds of spins with half-integer and integer spins alternatively. It is well-known that a half-integer antiferromagnetic spin chain has gapless excitation spectrum and the integer ones show the Haldane gap in their low energy spectrum, however the 1D ferrimagnets behave differently and exhibit intriguing quantum spin phases and thermodynamic properties. One of the intriguing features of the mixed-spin chain is that the ground state has a long-range ferrimagnetic order.

While the physics of the mixed-spin chain is well understood, the spin ladders are still under intensive investigation since they exhibit rich phase diagrams. So far, a large class of mixed-spin two-leg ladders are investigated, but most of studies concern the physics of unfrustrated mixed-spin ladders. For the unfrustrated bipartite ladder with alternating half-integer and integer spins on neighboring sites, the ground state is shown to exhibit long-range order [7]. In general, for the uniform spin ladders, frustration reduces the antiferromagnetic correlations and may produce various exotic quantum ground states such as the dimerized state [16–18]. In comparison with the uniform systems, less attention is paid to the role of frustration in the mixed spin systems and it is quite interesting to explore how the ferrimagnetic order is affected by the frustration [14,15]. Recently, the effect of magnetic frustrations due to diagonal exchange bonds in a mixed-spin ladder was studied by using the approach of exact numerical diagonalization [15] for a special case with equal strengths of the rung coupling and diagonal couplings. It was shown that the long-range ferrimagnetic state may be destroyed and a singlet state appears at a large value of the frustration parameter. However, it is not clear how the diagonal frustrations compete with the exchanges along the legs and whether an exotic intermediate phase exists for arbitrary rung coupling. Furthermore, an analytical analysis which works in both regimes with weak and strong frustrations is still lack. The general analytical method based on the spin wave theory is not capable to deal with the regime with strong frustration where the long-range order is destroyed. In order to provide a general understanding
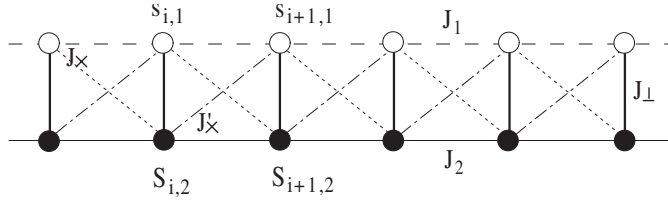
Fig. 1. Schematical representation of the mixed-spin ladder with diagonal exchanges. The open and close circles represent spins $s_{i,1} = 1/2$ and $S_{i,2} = 1$ respectively.

to frustration effect in the ferrimagnetic ladder system, in this work we carry out analytical investigation on the extended antiferromagnetic mixed-spin ladders in the limit of strong rung couplings. When the rung coupling is dominated, it is convenient and natural to map the original model onto an effective spin-1/2 exchange model. Based on the effective model, we show that the ground-state properties for both the frustrated and non-frustrated ladders can be understood within the same framework and the phase transition from a ferrimagnetic state to a critical spin-liquid state is also discussed. In our theory, we treat the weak and strong frustrations on same footing. For generality, we consider the most general frustrated ladder model with arbitrary strengths of exchanges of spins along the two legs as well as the diagonal exchanges. Our model can interpolate between a variety of systems, exhibiting remarkably rich ground state behavior with both ordered and disordered phases.

2 Model and the effective Hamiltonian

As shown in Figure 1, the extended frustrated mixed-spin ladder is made of two types of spins with magnitude $s_{i,1} = 1/2$ and $S_{i,2} = 1$ located on the rungs of the ladder. The corresponding Hamiltonian of the mixed-spin ladder takes the form

$$H = H_0 + H' \tag{1}$$

with

$$H_0 = \sum_{i=1}^{N} H_i = J_\perp \sum_{i=1}^{N} \widehat{s}_{i,1} \cdot \widehat{S}_{i,2} \tag{2}$$

and

$$H' = \sum_{i=1}^{N} H_{i,i+1} \tag{3}$$

where

$$H_{i,i+1} = H_{i,i+1}^{leg_1} + H_{i,i+1}^{leg_2} + H_{i,i+1}^{d_1} + H_{i,i+1}^{d_2} \tag{4}$$

and

$$\begin{aligned} H_{i,i+1}^{leg_1} &= J_1 \widehat{s}_{i,1} \cdot \widehat{s}_{i+1,1} \\ H_{i,i+1}^{leg_2} &= J_2 \widehat{S}_{i,2} \cdot \widehat{S}_{i+1,2} \\ H_{i,i+1}^{d_1} &= J_\times \widehat{s}_{i,1} \cdot \widehat{S}_{i+1,2} \\ H_{i,i+1}^{d_2} &= J_\times' \widehat{S}_{i,2} \cdot \widehat{s}_{i+1,1}. \end{aligned} \tag{5}$$

Here N is the number of rungs, and $\widehat{s}_{i,1}$ and $\widehat{S}_{i,2}$ represent the spin-1/2 and spin-1 operators respectively. For the antiferromagnetic ladder, all the coupling parameters $J_\perp, J_1, J_2, J_\times, J_\times' > 0$. The intrachain couplings along the up and down legs are denoted by J_1 and J_2 respectively. The interchain coupling across the rungs is $J_\perp$ and the diagonal exchanges between the rungs are $J_\times$ and $J_\times'$. The summation is over the length of the chains and the periodic boundary is assumed. For such a ladder model, the frustration comes from the competition between the intrachain couplings (J_1 and J_2) and the diagonal couplings ($J_\times$ and $J_\times'$). The model of (1) covers a variety of known models. The model with $J_1 = J_2 = 0$ corresponds to a bipartite mixed-spin ladder model whereas the one with $J_\times = J_\times' = 0$ is equivalent to the railroad ladder model. For $J_1 = J_2$ and $J_\times = J_\times'$, our model reduces to the ladder model investigated in reference [15], where only the case with $J_\perp = J_\times$ was considered. For $J_1 = J_2$ and $J_\times' = 0$, the model is the dimerized zigzag mixed-spin ladder [5, 14].

In this work, we will focus on the strong coupling limit with $J_\perp \gg J_1, J_2, J_\times, J_\times'$. In this limit, the interactions between the neighboring rungs can be treated as perturbations of the system of uncoupled rungs. It is instructive to start by considering the two-site problem on an isolated rung with the Hamiltonian given by

$$H_i = J_\perp \widehat{s}_{i,1} \cdot \widehat{S}_{i,2}.$$

Eigenstates of the local Hamiltonian on a rung can be classified according to the value of the total rung spin S_i. The two spins with magnitude $s_{i,1} = 1/2$ and $S_{i,2} = 1$ can combine into $S_i = 1/2$ and $3/2$. It is easy to get the eigenenergy $E_{1/2} = -J_\perp$ and $E_{3/2} = J_\perp/2$. The corresponding eigenstates are

$$\begin{aligned} \left| D_{\frac{1}{2}} \right\rangle_i &= \frac{1}{\sqrt{3}} \left[-\left| \frac{1}{2}, 0 \right\rangle_i + \sqrt{2} \left| -\frac{1}{2}, 1 \right\rangle_i \right], \\ \left| D_{-\frac{1}{2}} \right\rangle_i &= \frac{1}{\sqrt{3}} \left[\left| -\frac{1}{2}, 0 \right\rangle_i - \sqrt{2} \left| \frac{1}{2}, -1 \right\rangle_i \right] \end{aligned}$$

for the doublet and

$$\begin{aligned} \left| Q_{\frac{3}{2}} \right\rangle_i &= \left| \frac{1}{2}, 1 \right\rangle_i, \\ \left| Q_{\frac{1}{2}} \right\rangle_i &= \frac{1}{\sqrt{3}} \left[\sqrt{2} \left| +\frac{1}{2}, 0 \right\rangle_i + \left| -\frac{1}{2}, 1 \right\rangle_i \right], \\ \left| Q_{-\frac{1}{2}} \right\rangle_i &= \frac{1}{\sqrt{3}} \left[\sqrt{2} \left| -\frac{1}{2}, 0 \right\rangle_i + \left| \frac{1}{2}, -1 \right\rangle_i \right], \\ \left| Q_{-\frac{3}{2}} \right\rangle_i &= \left| -\frac{1}{2}, -1 \right\rangle_i \end{aligned}$$

for the quartet, where $|s_z, S_z\rangle_i = |s_z\rangle_{i,1} \otimes |S_z\rangle_{i,2}$. It is obvious that the ground state of a rung is a doublet and the excited state is a quartet with an excitation energy gap $3J_\perp/2$. Therefore in the strong rung-coupling limit spins on each rung of the ladder favor forming a doublet. Since each rung can be either in the doubly degenerate

state $\left|D_{\frac{1}{2}}\right\rangle$ or $\left|D_{-\frac{1}{2}}\right\rangle$, the ground state of the zero-order Hamiltonian H_0 is 2^N-foldly degenerate. When we go beyond the two-site problem, we need consider the inter-rung couplings. In general, the inter-rung exchanges will lift the degeneracy of the zero-order ground state and leads to an effective Hamiltonian acting on the groundstate Hilbert space of H_0.

We then derive the effective Hamiltonian of the original ladder model in the truncated Hilbert space composed of product of rung doublets by using perturbation method. Similar schemes have been applied to study uniform spin ladders in the strong-coupling limit [19,20]. To the first order and up to a constant of $-NJ_\perp$, the effective Hamiltonian can be represented as

$$H_{eff}^{(1)} = \sum_i \langle \mu_{i,i+1}| H_{i,i+1} |v_{i,i+1}\rangle |\mu_{i,i+1}\rangle \langle v_{i,i+1}| \quad (6)$$

where $\{|\mu_{i,i+1}\rangle, |v_{i,i+1}\rangle = \left|D_{\pm\frac{1}{2}}\right\rangle_i \otimes \left|D_{\pm\frac{1}{2}}\right\rangle_{i+1}\}$ are four-foldly degenerate. It is convenient to introduce pseudo-spin-1/2 operators $\widehat{\tau}_i$ which act on the states $\left|D_{1/2}\right\rangle_i$ and $\left|D_{-1/2}\right\rangle_i$ and are defined as

$$\begin{aligned}
\widehat{\tau}_i^z \left|D_{\pm\frac{1}{2}}\right\rangle_i &= \pm\frac{1}{2}\left|D_{\pm\frac{1}{2}}\right\rangle_i, \\
\widehat{\tau}_i^+ \left|D_{-\frac{1}{2}}\right\rangle_i &= \left|D_{+\frac{1}{2}}\right\rangle_i, \\
\widehat{\tau}_i^- \left|D_{+\frac{1}{2}}\right\rangle_i &= \left|D_{-\frac{1}{2}}\right\rangle_i, \\
\widehat{\tau}_i^+ \left|D_{+\frac{1}{2}}\right\rangle_i &= \widehat{\tau}_i^- \left|D_{-\frac{1}{2}}\right\rangle_i = 0.
\end{aligned}$$

With the above notation, we can identify the following relations

$$\begin{aligned}
\widehat{\tau}_i^z &= \frac{1}{2}\left|D_{+\frac{1}{2}}\right\rangle_{i\ i}\left\langle D_{+\frac{1}{2}}\right| - \frac{1}{2}\left|D_{-\frac{1}{2}}\right\rangle_{i\ i}\left\langle D_{-\frac{1}{2}}\right|, \\
\widehat{\tau}_i^+ &= \left|D_{+\frac{1}{2}}\right\rangle_{i\ i}\left\langle D_{-\frac{1}{2}}\right|, \\
\widehat{\tau}_i^- &= \left|D_{-\frac{1}{2}}\right\rangle_{i\ i}\left\langle D_{+\frac{1}{2}}\right|.
\end{aligned}$$

Therefore we can rewrite terms of $|\mu_{i,i+1}\rangle \langle v_{i,i+1}|$ in terms of the pseudo-spin operators. For example, we have

$$|\mu_{i,i+1}\rangle \langle v_{i,i+1}| = \widehat{\tau}_i^+ \widehat{\tau}_{i+1}^-$$

for $|\mu_{i,i+1}\rangle = \left|D_{+\frac{1}{2}}\right\rangle_i \left|D_{-\frac{1}{2}}\right\rangle_{i+1}$ and $|v_{i,i+1}\rangle = \left|D_{-\frac{1}{2}}\right\rangle_i \left|D_{+\frac{1}{2}}\right\rangle_{i+1}$ and the corresponding coefficient is given by

$$\langle \mu_{i,i+1}| H_{ij} |v_{i,i+1}\rangle = -\frac{2}{9}(J_\times + J_\times') + \frac{1}{18}J_1 + \frac{8}{9}J_2.$$

After some algebras, we can rewrite the effective Hamiltonian (6) as the following form

$$H_{eff}^{(1)}/J_\perp = \sum_i J_{eff}^{(1)} \left[\frac{1}{2}\left(\widehat{\tau}_i^+ \widehat{\tau}_{i+1}^- + \widehat{\tau}_i^- \widehat{\tau}_{i+1}^+\right) + \widehat{\tau}_i^z \widehat{\tau}_{i+1}^z\right] \quad (7)$$

with the effective parameter given by

$$J_{eff}^{(1)} = -\frac{4}{9}\left(\frac{J_\times}{J_\perp} + \frac{J_\times'}{J_\perp}\right) + \frac{1}{9}\frac{J_1}{J_\perp} + \frac{16}{9}\frac{J_2}{J_\perp}. \quad (8)$$

Next we carry out calculation of the second order perturbation which is expected to give correction to the effective coupling parameter and the ground energy. In the first order calculation, the higher excited states (quartets) on the isolated rungs do not play role. However, such rung quartets give contribution to the higher order correction. The second order correction can be described by

$$\begin{aligned}
H_{eff}^{(2)} = &\sum_{i,m\neq 0} \frac{\langle \mu_{i,i+1}| H_{i,i+1} |m\rangle \langle m| H_{i,i+1} |v_{i,i+1}\rangle}{E_0 - E_m} \\
&\times |\mu_{i,i+1}\rangle \langle v_{i,i+1}|
\end{aligned} \quad (9)$$

where $E_0 = 2E_{1/2} = -2J_\perp$ and $|m\rangle$ are the intermediate states with at least one quartet residing in the neighboring ith and $(i+1)$th rungs. Therefore $E_m = E_{1/2} + E_{3/2} = -J_\perp/2$ if $|m\rangle = \{|D_\alpha\rangle_i |Q_\beta\rangle_{i+1}$ or $|Q_\beta\rangle_i |D_\alpha\rangle_{i+1}\}$, and $E_m = 2E_{3/2} = J_\perp$ if $|m\rangle = |Q_\beta\rangle_i |Q_{\beta'}\rangle_{i+1}$ where $\alpha = -1/2, 1/2$, and $\beta, \beta' = -3/2, -1/2, 1/2, 3/2$. After tedious but straightforward calculation, we can represent $H_{eff}^{(2)}$ in terms of the pseudo-spin operators

$$H_{eff}^{(2)}/J_\perp = \sum_i \left[J_{eff}^{(2)}\widehat{\tau}_i \cdot \widehat{\tau}_{i+1} + c'\right], \quad (10)$$

where

$$\begin{aligned}
J_{eff}^{(2)} = &-\frac{4}{243}\left(\frac{J_1}{J_\perp} + 4\frac{J_2}{J_\perp} - 4\frac{J_\times}{J_\perp} - \frac{J_\times'}{J_\perp}\right)^2 \\
&-\frac{4}{243}\left(\frac{J_1}{J_\perp} + 4\frac{J_2}{J_\perp} - \frac{J_\times}{J_\perp} - 4\frac{J_\times'}{J_\perp}\right)^2 \\
&+\frac{8}{243}\left(\frac{J_1}{J_\perp} + \frac{J_2}{J_\perp} - \frac{J_\times}{J_\perp} - \frac{J_\times'}{J_\perp}\right)^2
\end{aligned} \quad (11)$$

and

$$\begin{aligned}
c' = &-\frac{4}{81}\left(\frac{J_1}{J_\perp} + \frac{J_2}{J_\perp} - \frac{J_\times}{J_\perp} - \frac{J_\times'}{J_\perp}\right)^2 \\
&-\frac{1}{81}\left(\frac{J_1}{J_\perp} + 4\frac{J_2}{J_\perp} - 4\frac{J_\times}{J_\perp} - \frac{J_\times'}{J_\perp}\right)^2 \\
&-\frac{1}{81}\left(\frac{J_1}{J_\perp} + 4\frac{J_2}{J_\perp} - \frac{J_\times}{J_\perp} - 4\frac{J_\times'}{J_\perp}\right)^2.
\end{aligned} \quad (12)$$

The second order correction can also produce a term due to the three-site process which is described by

$$\begin{aligned}
&\langle \mu_{i,i+1,i+2}| H_{eff}^{\prime(2)} |v_{i,i+1,i+2}\rangle = \\
&\sum_{i,m\neq 0} \left[\frac{\langle \mu_{i,i+1,i+2}| H_{i,i+1} |m\rangle \langle m| H_{i+1,i+2} |v_{i,i+1,i+2}\rangle}{E_0 - E_m}\right. \\
&\left. + \frac{\langle \mu_{i,i+1,i+2}| H_{i+1,i+2} |m\rangle \langle m| H_{i,i+1} |v_{i,i+1,i+2}\rangle}{E_0 - E_m}\right]
\end{aligned}$$

where $E_0 = 3E_{1/2} = -3J_\perp$ and $|m\rangle$ are the intermediate states with a quartet residing in the $(i+1)$th rung. In terms of the pseudo-spin operators, the Hamiltonian $H_{eff}^{'(2)}$ is finally simplified to

$$H_{eff}^{'(2)}/J_\perp = \sum_{i=1}^{N} J'_{eff} \widehat{\tau}_i \cdot \widehat{\tau}_{i+2} \qquad (13)$$

with

$$J'_{eff} = -\frac{8}{243}\left(\frac{J_1}{J_\perp} + 4\frac{J_2}{J_\perp} - \frac{J_\times}{J_\perp} - 4\frac{J'_\times}{J_\perp}\right)$$
$$\times \left(\frac{J_1}{J_\perp} + 4\frac{J_2}{J_\perp} - 4\frac{J_\times}{J_\perp} - \frac{J'_\times}{J_\perp}\right). \qquad (14)$$

Therefore, up to the second order, the effective Hamiltonian of the original ladder can be written as

$$H_{eff}/J_\perp = \sum_{i=1}^{N} \left(J_{eff} \widehat{\tau}_i \cdot \widehat{\tau}_{i+1} + J'_{eff} \widehat{\tau}_i \cdot \widehat{\tau}_{i+2} + c\right), \qquad (15)$$

where $J_{eff} = J_{eff}^{(1)} + J_{eff}^{(2)}$ and $c = -1 + c'$. The effective Hamiltonian (15) describes a spin-1/2 Heisenberg chain with additional next-nearest-neighbor (NNN) exchanges.

3 Phase diagram and discussion

It is convenient to determine the phase diagram of the original ladder model from the effective Hamiltonian (15) since it has a much simpler form and was widely studied. Since the terms with NNN exchanges come from the second-order perturbation, in general we always have $\left|J'_{eff}\right| \ll |J_{eff}|$. A small NNN ferromagnetic interaction is an irrelevant perturbation to the spin chain model [17,21]. Therefore we may approximate to determine the phase diagram by omitting the term of J'_{eff}. The effective Hamiltonian (15) with $J'_{eff} = 0$ describes either a ferromagnetic or an antiferromagnetic spin-1/2 Heisenberg chain depending on the sign of the effective parameter. Based on the effective Hamiltonian, one is ready to get the ground energy of the mixed-spin ladder (1) in the strong coupling limit:

$$E_g/NJ_\perp = \varepsilon_H + \delta\varepsilon' + c, \qquad (16)$$

where ε_H is the ground energy per site of an isotropic spin-1/2 Heisenberg chain and $\delta\varepsilon'$ is the energy correction due to the NNN exchanges which reads

$$\delta\varepsilon' = \langle GS| H_{eff}^{'(2)} |GS\rangle /NJ_\perp$$

with $|GS\rangle$ representing the ground state of Heisenberg chain. From the corresponding results of the well-known Heisenberg chain [22,23], we have

$$\varepsilon_H/J_{eff} = \begin{cases} 1/4, & \text{for } J_{eff} < 0 \\ 1/4 - \ln 2, & \text{for } J_{eff} > 0. \end{cases}$$

When the system is in the ferromagnetic state, it is ready to obtain

$$\delta\varepsilon' = J'_{eff}/4.$$

When the system is in the antiferromagnetic phase, no an analytical result is available.

From equation (8), we see that the diagonal exchanges tend to produce an effective ferromagnetic coupling whereas the exchanges along the legs lead to an effective antiferromagnetic coupling. Both the ferromagnetic and antiferromagnetic spin-1/2 Heisenberg chains are exactly solved by the Bethe-ansatz method [22,23], but they have very different ground-state properties. The ferromagnetic ground state consists of all spins parallel and thus has long-range order, whereas the antiferromagnetic ground state is a spin singlet which can be described by the critical spin-liquid phase. If $J_{eff} < 0$, the effective model is a ferromagnetic chain which exhibits gapless excitations and ferromagnetic ground state. Equivalently, within the first order approximation the original mixed-spin ladder has ferrimagnetic ground state if $J_\times + J'_\times > J_1/4 + 4J_2$. On the other hand, the effective model is an antiferromagnetic chain if $J_{eff} > 0$. Correspondingly, the ground state of the original mixed-spin ladder is a critical spin liquid if $J_\times + J'_\times < J_1/4 + 4J_2$.

As we have mentioned that our mixed-spin ladder model includes a series of models as its special cases, we may understand the ground-state properties of these models by using our derived effective Hamiltonian (15). When $J_1 = J_2 = 0$, our model reduces to a bipartite mixed-spin ladder model. The effective model (15) is a ferromagnetic Heisenberg chain because we have $J_{eff} < 0$. Therefore we can conclude that the total spin of the ground state is $N/2$. When $J_\times = J'_\times = 0$, we get the railroad ladder model composed of coupled spin-1/2 and spin-1 chains. It is ready to conclude that its ground state is a singlet because the effective model (15) is an antiferromagnetic Heisenberg chain with $J_{eff} > 0$. For the frustrated spin ladder with $J_1 = J_2$ and $J_\times = J'_\times$, it is easy to see from the effective model (15) that there exists a quantum phase transition arising from the competition of J_1 and $J_\times$. The transition point is determined by $J_{eff} = 0$ with

$$J_{eff} = -\frac{8}{9}\frac{J_\times}{J_\perp} + \frac{17}{9}\frac{J_1}{J_\perp} - \frac{56}{81}\left(\frac{J_1}{J_\perp} - \frac{J_\times}{J_\perp}\right)^2. \qquad (17)$$

From the solution of the above equation, we can get the phase diagram of the mixed spin ladder as shown in Figure 2. Above the phase boundary the original ladder is in a ferrimagnetic phase corresponding to $J_{eff} < 0$, whereas the system is in an antiferromagnetic phase with $J_{eff} > 0$ below the phase boundary.

In order to check how good is our theory based on the perturbation method and the validation of the phase boundary determined by $J_{eff} = 0$, we study the original spin ladder model by exact diagonalization method and determine its phase diagram numerically. The phase boundary determined by the effective spin chain model is not sensitive to the size of the system (Actually the phase boundary is completely independent of the size when we
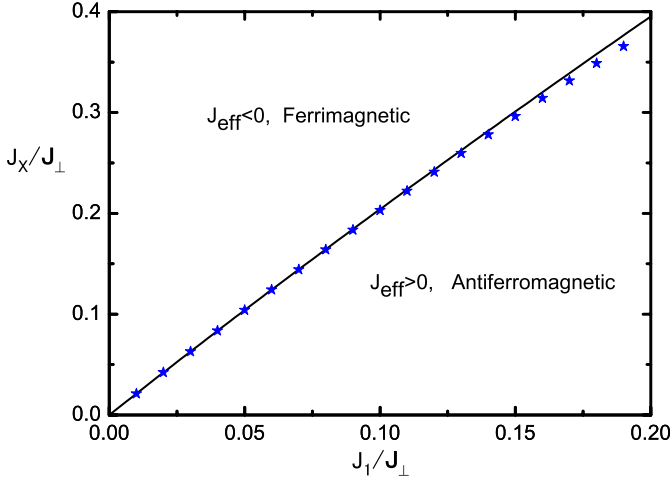
Fig. 2. Phase diagram of the mixed-spin ladder with $J_1 = J_2$ and $J_\times = J'_\times$ in the strong coupling limit. The solid line is the phase boundary determined by $J_{eff} = 0$ whereas the separated dots denoted by stars are obtained from the exact diagonalization of the spin ladder model.

omit the term of NNN interaction). Therefore for the purpose of verification of our analytical result, it is enough for us to consider a 2×4-size ladder which can be diagonalized by the exact diagonalization method. The phase boundary can be determined simply by the ground state degeneracy of the system. In comparison with phase boundary obtained by omitting the term of the NNN exchanges, we find that they agree very well in the strong coupling limit and as expected, the analytical result begins to deviate the exact numerical result when $J_1/J_\perp$ increases.

Finally, we address a special point with $J_1 = J_2 = J_\times = J'_\times$ where the total spin on a rung is a good quantum number. From equation (8), it is obvious that the model is effectively described by an antiferromagnetic spin chain with an effective coupling constant of J_1 and the high-order corrections vanish due to $J_{eff}^{(2)} = J'_{eff} = 0$. Actually this is an exact conclusion for this special case. This is clear if we rewrite the original Hamiltonian as

$$H = \sum_{i=1}^{N} \left[\frac{J_\perp}{2} \widehat{\mathbf{S}}_i^2 + J_1 \widehat{\mathbf{S}}_i \cdot \widehat{\mathbf{S}}_{i+1} - \frac{11 J_\perp}{8} \right], \qquad (18)$$

where $\widehat{\mathbf{S}}_i^2 = S_i(S_i + 1)$ with $S_i = 1/2$ or $3/2$ and $\widehat{\mathbf{S}}_i$ is defined as $\widehat{\mathbf{S}}_i \equiv \widehat{s}_{i,1} + \widehat{S}_{i,2}$ denoting the total spin on the ith rung. In the strong coupling limit $J_\perp \gg J_1, J_\perp$ forces S_i to take the value of $1/2$ and thus the ground-state properties of H is described by an effective spin-1/2 antiferromagnetic Heisenberg chain. From equations (11) and (12), we see that $J_{eff}^{(2)} = c' = 0$ when $J_1 = J_2 = J_\times = J'_\times$. This implies that the higher order correction also conforms to the exact result in this special case.

4 Conclusions

In conclusion, we have studied the ground-state properties of a generalized mixed-spin ladder with diagonal exchanges in the limit of strong rung couplings. By mapping it to an effective spin-1/2 Heisenberg chain with additional NNN exchanges, we find that the diagonal exchanges lead to the ferromagnetic effective coupling whereas the exchanges along the legs produces the antiferromagnetic effective coupling. The ground state of the effective Hamiltonian is either ferromagnetic or antiferromagnetic depending on the competition between these two opposite processes. With the help of the effective Hamiltonian and omitting the additional small NNN exchanges, it is straightforward to analytically determine the transition point from the ferrimagnetic phase to the critical spin liquid phase. The phase boundary is found to agree with the result obtained by exact numerical diagonalization of the original spin ladder model. Our results show that the strong coupling approach provides a simple and unifying way to exhibit the rich physics of the mixed-spin ladders.

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